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# Efficient Stress Solutions at Skin Stiffener Interfaces of Composite Stiffened Panels

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**Solutions for the stresses at the skin stiffener interface of composite stiffened panels are developed. Initially, the governing partial differential equations for the stresses are set up and the solutions are found. These solutions are then used in conjunction with an energy minimization approach to determine unknown constants in the analyticaly determined stress expressions. Results of this method are compared to boundary element solutions and are shown to be in good agreement. The present method has the advantage of being in closed form and very efficient. It can therefore be used to screen design candidates and give an accurate idea of the stress field near dropped plies without resorting to time-consuming finite element or other solutions.**

## Nomenclature

| | |
|---|---|
| $A_1, C_n, D_n$ | = constants in stress expressions |
| $E_0$, $E_1$ | = constants describing far-field stress dependence |
| $f(x), F_1(x), F_2(x)$ | = functions describing in-plane stress dependence |
| $g(z)$ | = function describing out-of-plane stress dependence |
| $n$ | = index denoting term in the series |
| $S_{ij}$ | = compliances in each layer, $i,j = 1,2, \ldots 6$ |
| $t_1, t_2$ | = thickness of regions 1 and 2 |
| $x$ | = in-plane coordinate perpendicular to stiffener axis |
| $z$ | = out-of-plane coordinate perpendicular to the skin |
| $\beta, \gamma$ | = constants in the governing partial differential equations |
| $\gamma_{ij}, \epsilon_i$ | = shear and normal strains, $i,j = x,y,z$ |
| $\lambda$ | = constant describing out-of-plane dependence; characteristic value |
| $\Pi_c$ | = complementary energy |
| $\sigma_i, \tau_{ij}$ | = normal and shear stresses, $i,j = x,y,z$ |
| $\phi_1, \phi_2$ | = exponents in-plane stress dependence; solutions to the characteristic equation |

## Introduction

As composites are increasingly used on primary structures of aircraft, the issue of developing efficient stress solutions that will allow accurate design, tradeoff studies, and failure predictions becomes of primary importance. A general approach is being developed that can predict the full three-dimensional stress field for a variety of boundary-layer problems in which one dimension is long compared to the other two. A sample of problems under consideration is shown in Fig. 1. Two of these problems (skin stiffener interaction and external ply drop) are addressed in this paper.

One of the common failure modes for composite stiffened structures is that of skin stiffener separation, which results from high localized interlaminar stresses at the flange termination. This paper presents a simple efficient closed-form solution to determine the stress field near the flange termination as a function of loads applied far from the termination point that are assumed to be known from other methods (for example, coarse finite elements model of the structure). The geometry and coordinate system for the problem are shown in Fig. 2.

Solutions to this problem have been presented in the past by various authors, in particular Hyer and Cohen,[1,2] where the stress function approach developed by Lekhnitskii[3] and applied by Wang and Choi[4,5] to problems with a free edge was used. In their second paper, Cohen and Hyer[2] included the effect of geometric nonlinearities. More recently, an approach very similar to that suggested by Hyer and Cohen has been developed by Kan et al.[6] Some experimental results for pressure loaded panels were presented by Hyer et al.[7] The results showed that the two interlaminar shear stresses contribute significantly to the stress field and the peel stress is not solely responsible for skin/stiffener separation.

The analysis methods just mentioned, although very accurate and reliable, involve complex eigenvalue solutions for the stress singularity at the flange edge and the use of complex variables. As a result, they are time consuming and do not lend themselves to use by designers at the early stages of a program where trade studies are performed to evaluate various designs. Other solutions based on finite elements are equally time consuming, requiring very fine meshes near the flange termination. In what follows, a stress-based solution method is presented that, without losing much of the accuracy of the more detailed solutions, can be used to gain insight into which problem parameters are important and how a design can be modified to avoid skin stiffener separation at low applied loads. The method will be formulated for the general three-dimensional case, and a two-dimensional example problem will be investigated.

## Governing Equations

The governing equations are derived assuming that stresses and strains do not depend on $y$ (the stiffener axial direction). For convenience, each layer is assumed to be balanced so the compliances $S_{16}$, $S_{26}$, $S_{36}$, and $S_{45}$ are zero. This assumption uncouples the governing equations for the stresses in each layer. If this assumption is relaxed, the governing equations are coupled. The structure of Fig. 2 is divided into layers, each corresponding to a single ply (if it is balanced) or consisting of more than one ply. The constitutive law for each layer is given in terms of the compliances $S_{ij}$ ($i,j = 1,2, \ldots 6$). These compliances will, in general, be different from layer to layer.

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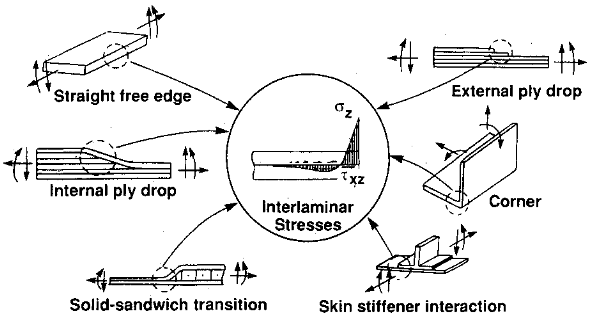


**Fig. 1 Boundary-layer problems in composites.**

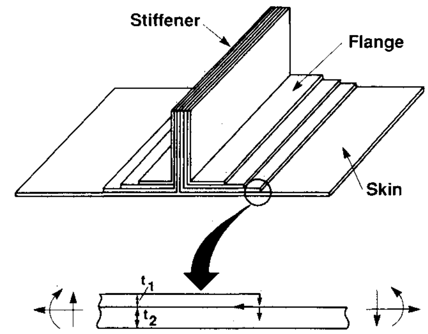


**Fig. 2 Skin stiffener geometry.**

The strain compatibility equations in each layer have the form[8]:

$$\frac{\partial^2\gamma_{yz}}{\partial y\partial z}=\frac{\partial^2\epsilon_y}{\partial z^2}+\frac{\partial^2\epsilon_z}{\partial y^2} \tag{1a}$$

$$\frac{\partial^2\gamma_{xy}}{\partial x\partial y}=\frac{\partial^2\epsilon_x}{\partial y^2}+\frac{\partial^2\epsilon_y}{\partial x^2} \tag{1b}$$

$$\frac{\partial^2\gamma_{xz}}{\partial x\partial z}=\frac{\partial^2\epsilon_x}{\partial z^2}+\frac{\partial^2\epsilon_z}{\partial x^2} \tag{1c}$$

$$2\frac{\partial^2\epsilon_x}{\partial y\partial z}=\frac{\partial}{\partial x}\left(\frac{-\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{xz}}{\partial y}+\frac{\partial\gamma_{xy}}{\partial z}\right) \tag{1d}$$

$$2\frac{\partial^2\epsilon_z}{\partial x\partial y}=\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{xz}}{\partial y}-\frac{\partial\gamma_{xy}}{\partial z}\right) \tag{1e}$$

$$2\frac{\partial^2\epsilon_y}{\partial x\partial z}=\frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{xz}}{\partial y}+\frac{\partial\gamma_{yz}}{\partial z}\right) \tag{1f}$$

and using the fact that strains do not depend on $y$, they can be simplified to

$$\frac{\partial^2\epsilon_y}{\partial z^2}=0 \tag{2a}$$

$$\frac{\partial^2\epsilon_y}{\partial x^2}=0 \tag{2b}$$

$$\frac{\partial^2\gamma_{xz}}{\partial x\partial z}=\frac{\partial^2\epsilon_x}{\partial z^2}+\frac{\partial^2\epsilon_z}{\partial x^2} \tag{2c}$$

$$\frac{\partial}{\partial x}\left(\frac{-\partial\gamma_{yz}}{\partial x}+\frac{\partial\gamma_{xy}}{\partial z}\right)=0 \tag{2d}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial\gamma_{yz}}{\partial x}-\frac{\partial\gamma_{xy}}{\partial z}\right)=0 \tag{2e}$$

$$\frac{\partial^2\epsilon_y}{\partial x\partial z}=0 \tag{2f}$$

The equilibrium equations reduce to the form:

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{xz}}{\partial z}=0 \tag{3a}$$

$$\frac{\partial\tau_{xy}}{\partial x}+\frac{\partial\tau_{yz}}{\partial z}=0 \tag{3b}$$

$$\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\sigma_z}{\partial z}=0 \tag{3c}$$

Finally, the strains are expressed in terms of stresses through the constitutive law:

$$\begin{Bmatrix}\epsilon_x\\ \epsilon_y\\ \epsilon_z\\ \gamma_{yz}\\ \gamma_{xz}\\ \gamma_{xy}\end{Bmatrix}=\begin{bmatrix}S_{11}&S_{12}&S_{13}&0&0&0\\ S_{12}&S_{22}&S_{23}&0&0&0\\ S_{13}&S_{23}&S_{33}&0&0&0\\ 0&0&0&S_{44}&0&0\\ 0&0&0&0&S_{55}&0\\ 0&0&0&0&0&S_{66}\end{bmatrix}\begin{Bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{yz}\\ \tau_{xz}\\ \tau_{xy}\end{Bmatrix} \quad \begin{matrix}(4a)\\ (4b)\\ (4c)\\ (4d)\\ (4e)\\ (4f)\end{matrix}$$

Using the strain-stress Eqs. (4) to substitute in Eqs. (2a) and (2b), the following expressions are derived:

$$S_{12}\frac{\partial^2\sigma_x}{\partial z^2}+S_{22}\frac{\partial^2\sigma_y}{\partial z^2}+S_{23}\frac{\partial^2\sigma_z}{\partial z^2}=0 \tag{5a}$$

$$S_{12}\frac{\partial^2\sigma_x}{\partial x^2}+S_{22}\frac{\partial^2\sigma_y}{\partial x^2}+S_{23}\frac{\partial^2\sigma_z}{\partial x^2}=0 \tag{5b}$$

These two equations are compatible with each other (and the assumption that stresses do not depend on $y$) only if

$$\sigma_y=k_0+k_1x+k_2z+k_3xz-\frac{S_{12}}{S_{22}}\sigma_x-\frac{S_{23}}{S_{22}}\sigma_z \tag{6}$$

which expresses the $\sigma_y$ stress as a function of $\sigma_x$ and $\sigma_z$. The constants $k_0$, $k_1$, $k_2$, and $k_3$ are determined from boundary conditions. From the equilibrium equations (3) the following expressions are obtained

$$\frac{\partial^2\sigma_z}{\partial z^2}=\frac{-\partial^2\tau_{xz}}{\partial x\partial z}=\frac{\partial^2\sigma_x}{\partial x^2} \tag{7a}$$

$$\frac{\partial^4\sigma_z}{\partial x^2\partial z^2}=\frac{-\partial^4\tau_{xz}}{\partial x^3\partial z}=\frac{\partial^4\sigma_x}{\partial x^4} \tag{7b}$$

$$\frac{\partial^2\tau_{yz}}{\partial x\partial z}=\frac{-\partial^2\tau_{xy}}{\partial x^2} \tag{7c}$$

Equations (6) and (7) are used to eliminate $\sigma z$ and $\tau xz$ from Eq. (2c):

$$\left(S_{55}+2S_{13}-\frac{S_{23}S_{12}}{S_{22}}\right)\frac{\partial^4\sigma_x}{\partial x^2\partial z^2}+\left(S_{11}-\frac{S_{12}^2}{S_{22}}\right)\frac{\partial^4\sigma_x}{\partial z^4}$$
$$\left.-\frac{S_{12}S_{23}}{S_{22}}\right)\frac{\partial^4\sigma_x}{\partial x^2\partial z^2}+\left(S_{33}-\frac{S_{23}^2}{S_{22}}\right)\frac{\partial^4\sigma_x}{\partial x^4}=0 \tag{8}$$

which by letting $\beta=(S_{55}\ S_{22}\ +2\ S_{13}\ S_{22}\ -2\ S_{12}\ S_{23})/(S_{33}\ S_{22}\ -S_{23}{}^2)$ and $\gamma=(S_{11}\ S_{22}\ -S_{12}{}^2)/(S_{33}\ S_{22}\ -S_{23}{}^2)$ is rewritten:

$$\frac{\partial^4\sigma_x}{\partial x^4}+\beta\frac{\partial^4\sigma_x}{\partial x^2\partial z^2}+\gamma\frac{\partial^4\sigma_x}{\partial z^4}=0 \tag{9}$$

The remaining equation for $\tau xy$ is obtained by using Eqs. (2d) and (2e). [Note that the remaining Eq. (2f) is, in view of the assumptions and previous equations, merely a statement of the fact that the transverse in-plane deflection $v$ is linear in $y$.] Equations (2d) and (2e) are compatible with each other only if

$$\frac{\partial \gamma_{yz}}{\partial x} = C_0 - \frac{\partial \gamma_{xy}}{\partial z} \tag{10}$$

where $C_0$ is an unknown constant. Using the strain-stress equations (4d) and (4f) to express the strains in terms of stresses and the last of Eqs. (7) to eliminate $\tau_{yz}$, the governing equation for $\tau_{xy}$ is obtained:

$$\frac{\partial^2 \tau_{xy}}{\partial x^2} + \frac{S_{66}}{S_{44}} \frac{\partial^2 \tau_{xy}}{\partial z^2} = 0 \tag{11}$$

Equations (9) and (11) are the governing equations of the problem. The advantage of this approach is that no stress functions are needed. These equations are to be solved in each layer subject to the stress boundary conditions and stress continuity at the interface between layers.

## Solution

In this paper, a solution for Eq. (9), the more complicated of the two governing equations, will be presented. Far from the flange edge ($x$ large), the interlaminar stresses are assumed to decay to zero and the in-plane stresses are known functions of $z$, which describe the far-field applied loads. Here, the far-field dependence of $\sigma_x$ is assumed to be linear in $z$, which would correspond to a case of tension, compression, and/or bending moment. As will be shown, the far-field solution can be any polynomial expression in $z$, and the solution can be easily modified for that expression.

Solutions to Eq. (9) are sought in the form of linear combinations of products of functions of $x$ and $z$, i.e.,

$$\sigma_x \propto f(x)g(z) \tag{12}$$

By judiciously selecting function forms for $g$, Eq. (9) becomes an ordinary differential equation in $f$. Letting $g = \sin \lambda_z$ and substituting in Eq. (9), the following equation is obtained for $f$:

$$\frac{d^4 f}{dx^4} - \beta\lambda^2 \frac{d^2 f}{dx^2} + \gamma\lambda^4 f = 0 \tag{13}$$

which has the solution

$$f = C_1 e^{-\phi_1 x} + C_2 e^{-\phi_2 x} + C_3 e^{-\phi_1 x} + C_4 e^{-\phi_2 x} \tag{14}$$

where $\phi_1$ and $\phi_2$ are the solutions of the characteristic equation

$$\phi^4 - \beta\lambda^2\phi^2 + \gamma\lambda^4 = 0 \tag{15}$$

and thus,

$$\phi_1 = \frac{\lambda}{\sqrt{2}} \left(\beta + \sqrt{\beta^2 - 4\gamma}\right)^{1/2}$$
$$\phi_2 = \frac{\lambda}{\sqrt{2}} \left(\beta - \sqrt{\beta^2 - 4\gamma}\right)^{1/2} \tag{16}$$

To avoid interlaminar stresses that grow indefinitely with distance from the flange edge ($x$ large), the constants $C_3$ and $C_4$ are set equal to zero. A similar solution can be obtained for $g = \cos \mu_z$. The constants $\lambda$ and $\mu$ are chosen so as to satisfy some of the stress boundary conditions of the problem at $z = 0$ and $z = t_2$. For simplicity, the problem of Fig. 2 is assumed to be split in two layers, the dropped region (region 1) and the portion of the continuous region that is directly below it (i.e., up to $x = 0$). The latter will be region 2. Note that, in each region, $x$ starts at the edge of the flange and is positive to the left (Fig. 2). Also, the origin of the $z$ axis shifts from region to region so that, in region 1, $0 < z < t_1$ and in region 2, $0 < z < t_2$ with $z = 0$ at the top of each region.

For this problem, $\lambda$ and $\mu$ are selected to equal $n\pi/t_1$. Then, the solution to Eq. (9) in region 1 has the form

$$\sigma_x = \sum_{n=1}^{\infty} \left(A_1 e^{-\phi_1 x} + e^{-\phi_2 x}\right)\left(C_n \sin \frac{n\pi z}{t_1} + D_n \cos \frac{n\pi z}{t_1}\right) + E_0 + E_1 z \tag{17}$$

where the summation is over $n$. $A_1$, $C_n$, and $D_n$, are unknown constants to be determined from boundary conditions. The terms involving exponentials decay to zero for large $x$ and the linear stress distribution (assumed known) $\sigma_x = E_0 + E_1 z$ at the far field is recovered.

Since $\phi_1$ and $\phi_2$ depend on $t_1$, displacement and/or stress continuity at the interface between regions 1 and 2 may not be satisfied in all instances. A discussion of this limitation and a way to satisfy continuity at that interface is given after the stress expressions in region 2 are developed.

Using the equilibrium equations (3) and the boundary conditions that require $\sigma_z = \tau_{xz} = 0$ at $z = 0$ (top of region 1), and $\tau_{xz} = 0$ at $x = 0$, the expressions for $\sigma_x$, $\tau_{xz}$, and $\sigma_z$ are obtained in region 1.

$$\sigma_x = \sum_1^{\infty}\left(-\frac{\phi_2}{\phi_1} e^{-\phi_1 x} + e^{-\phi_2 x}\right)\left(C_n \sin \frac{n\pi z}{t_1} + D_n \cos \frac{n\pi z}{t_1}\right) + E_0 + E_1 z$$

$$\tau_{xz} = \sum_1^{\infty} (e^{-\phi_2 x} - e^{-\phi_1 x})\phi_2 \frac{t_1}{n\pi}\left[C_n\left(1 - \cos \frac{n\pi z}{t_1}\right) + D_n \sin \frac{n\pi z}{t_1}\right]$$

$$\sigma_z = \sum_1^{\infty}\left(\phi_2 e^{-\phi_2 x} - \phi_1 e^{-\phi_1 x}\right)\phi_2 \left(\frac{t_1}{n\pi}\right)^2 \left[C_n\left(\frac{n\pi z}{t_1} - \sin \frac{n\pi z}{t_1}\right) + D_n\left(1 - \cos \frac{n\pi z}{t_1}\right)\right] \tag{18}$$

where $C_n$ and $D_n$ are still unknown. Equations (18) satisfy equilibrium even if $\phi_1$ and $\phi_2$ are treated as unknown constants. In a similar manner, using Eq. (17), the equilibrium conditions (3a-c) and the boundary conditions ($\sigma_z = \tau_{xz} = 0$ at $z = t_2$) the following expressions are obtained for the stresses in region 2:

$$\sigma_x = \sum_1^{\infty}\left(\bar{A}_1 e^{-\phi_1 x} + e^{-\phi_2 x}\right)\left(\bar{C}_n \sin \frac{n\pi z}{t_2} + D_n \cos \frac{n\pi z}{t_2}\right) + \bar{E}_0 + \bar{E}_1 z$$

$$\tau_{xz} = \sum_1^{\infty}\left(\phi_1 \bar{A}_1{}^{-\phi_2 x} + \phi_2 e^{-\phi_1 x}\right)\frac{t_2}{n\pi}\left[\bar{C}_n\left(1 - \cos \frac{n\pi z}{t_2}\right) + \bar{D}_n \sin \frac{n\pi z}{t_2}\right] + F_1(x)$$

$$\sigma_z = \sum_1^{\infty}\left(\bar{A}_1 \phi_1{}^2 e^{-\phi_1 x} - \phi_2{}^2 e^{-\phi_2 x}\right)\frac{t_2^2}{n\pi}\left[\bar{C}_n\left(\frac{n\pi z}{t_2} - \sin \frac{n\pi z}{t_2}\right) + \bar{D}_n\left(1 - \cos \frac{n\pi z}{t_2}\right)\right] + F_1(x)z + F_2(x) \tag{19}$$

where $\bar{E}_0$ and $\bar{E}_1$ are constants that describe the far-field variation of $\sigma_x$ in region 2 and are assumed known. The unknown functions $F_1(x)$ and $F_2(x)$ are determined by requiring that $\sigma_z$

and $\tau_{xz}$ are continuous across the interface between regions 1 and 2:

$$\tau_{xz}^{(1)}(z=t_1)=\tau_{xz}^{(2)}(z=0)$$
$$\sigma_z^{(1)}(z=t_1)=\sigma_z^{(2)}(z=0) \tag{20}$$

In addition, using Eqs. (20), the unknowns $\bar{A}_1$, $\bar{C}_n$, and $\bar{D}_n$ in region 2 can be determined for $n$ odd. The resulting expressions are

$$\bar{A}_1=-\frac{\phi_2}{\phi_1} \tag{21a}$$

$$\bar{C}_n=-C_n\frac{t_1}{t_2} \tag{21b}$$

$$\bar{D}_n=-\frac{n\pi t_1(t_1+t_2)}{2t_2^2}-\frac{t_1^2}{t_2^2}D_n \tag{21c}$$

Instead of extending the solution into region 3 (for $x<0$ in Fig. 2), the force equilibrium conditions at $x=0$ in region 2 are imposed:

$$\int_0^{t_2}\tau_{xz}\,\mathrm{d}z\,\bigg|_{x=0}=0 \tag{22a}$$

$$\int_0^{t_2}\sigma_x\,\mathrm{d}z\,\bigg|_{x=0}=E_0t_1+E_1\frac{t_1^2}{2} \tag{22b}$$

Equation (22a) is identically satisfied. Equation (22b) has the form

$$-\sum_1^\infty\left(-\frac{\phi_2}{\phi_1}+1\right)\left[\bar{C}_n\frac{t_2}{n\pi}(\cos n\pi-1)\right]=E_0t_1+\frac{E_1t_1^2}{2}$$

which using Eq. (21b) becomes

$$\sum_1^\infty-2\left(\frac{\phi_1-\phi_2}{\phi_1}\right)\frac{1}{\pi}\frac{C_n}{n}=E_0+\frac{E_1t_1}{2}=\frac{1}{t_1}\int_0^{t_1}(E_0+E_1z)\,\mathrm{d}z$$

By expanding the right-hand side into a Fourier series valid for $n$ odd in the region $0<z<t_2$

$$\sum_1^\infty-2\left(\frac{\phi_1-\phi_2}{\phi_1}\right)\frac{1}{\pi}\frac{C_n}{n}=\frac{4}{\pi^2}\sum_1^\infty\frac{\cos n\pi-1}{n^2}\left(E_0+\frac{E_1t_1}{2}\right)$$

from which $C_n$ is determined as

$$C_n=-\frac{4}{n\pi}\left(E_0+\frac{E_1t_1}{2}\right)\frac{\phi_1}{\phi_1-\phi_2},\qquad n\text{ odd} \tag{23}$$

The condition that the flange edge is stress free in region 1 ($x=0$) is now imposed:

$$\sigma_x^{(1)}(x=0)=0$$

which implies

$$\sum_1^\infty\left(-\frac{\phi_2}{\phi_1}+1\right)\left[-\frac{4}{n\pi}\left(E_0+\frac{E_1t_1}{2}\right)\frac{\phi_1}{\phi_1-\phi_2}+D_n\cos\frac{n\pi z}{t_1}\right]+E_0+E_1z=0 \tag{24}$$

This can be satisfied by expanding $E_0+E_1z$ in a Fourier series:

$$E_0+E_1z=-\frac{4}{\pi^2}\sum_1^\infty\left(\frac{\cos n\pi-1}{n^2}\right)\left(E_0+\frac{E_1t_1}{2}\right)\sin\frac{n\pi z}{t_1}\times\sum_1^\infty\frac{2E_1t_1}{n^2\pi^2}(\cos n\pi-1)\cos\frac{n\pi z}{t_1} \tag{25}$$

Comparing Eqs. (24) and (25), $D_n$ is obtained as

$$D_n=-\frac{2E_1t_1}{n^2\pi^2}\frac{\phi_1}{\phi_1-\phi_2}(\cos n\pi-1),\qquad n\text{ odd} \tag{26}$$

valid for $n$ odd. Therefore, since all conditions are now satisfied using only odd $n$, the part of the series involving even $n$ is not needed. The final expressions for the stresses in regions 1 and 2 are

Region 1:

$$\sigma_x=\sum_1^\infty\left(-\frac{\phi_2}{\phi_1}e^{-\phi_1x}+e^{-\phi_2x}\right)\frac{4}{\pi}\frac{\phi_1}{\phi_1-\phi_2}\times\left[-\left(E_0+\frac{E_1t_1}{2}\right)\frac{1}{n}\sin\frac{n\pi z}{t_1}+\frac{E_1t_1}{n^2\pi}\cos\frac{n\pi z}{t_1}\right]+E_0+E_1z$$

$$\tau_{xz}=\sum_1^\infty\left(e^{-\phi_2x}-e^{-\phi_1x}\right)\frac{4}{\pi^2}\frac{\phi_1\phi_2}{\phi_1-\phi_2}t_1\times\left[-\left(E_0+\frac{E_1t_1}{2}\right)\frac{1}{n^2}\left(1-\cos\frac{n\pi z}{t_1}\right)+\frac{E_1t_1}{\pi n^3}\sin\frac{n\pi z}{t_1}\right]$$

$$\sigma_z=\sum_1^\infty\left(\phi_2e^{-\phi_2x}-\phi_1e^{-\phi_1x}\right)\frac{\phi_1\phi_2}{\phi_1-\phi_2}\frac{4t_1^2}{\pi^3}\times\left[-\left(E_0+\frac{E_1t_1}{2}\right)\frac{1}{n^3}\left(\frac{n\pi z}{t_1}-\sin\frac{n\pi z}{t_1}\right)+\frac{E_1t_1}{n^4\pi}\left(1-\cos\frac{n\pi z}{t_1}\right)\right] \tag{27a}$$

Region 2:

$$\sigma_x=\sum_1^\infty\left(-\frac{\phi_2}{\phi_1}e^{-\phi_1x}+e^{-\phi_2x}\right)\frac{4t_1}{t_2}\frac{\phi_1}{\phi_1-\phi_2}\times\left\{\frac{1}{n\pi}\left(E_0+\frac{E_1t_1}{2}\right)\sin\frac{n\pi z}{t_2}+\frac{1}{t_2}\left[\frac{t_1+t_2}{2}\left(E_0+\frac{E_1t_1}{2}\right)-\frac{t_1}{n^2\pi^2}E_1t_1\right]\cos\frac{n\pi z}{t_2}\right\}+\bar{E}_0+\bar{E}_1z$$

$$\tau_{xz}=\sum_1^\infty\left(e^{-\phi_2x}-e^{-\phi_1x}\right)\frac{4t_1}{n\pi}\frac{\phi_1\phi_2}{\phi_1-\phi_2}\left\{\frac{1}{t_2}\left[\frac{t_1+t_2}{2}\times\left(E_0+\frac{E_1t_1}{2}\right)-\frac{t_1E_1t_1}{n^2\pi^2}\right]\sin\frac{n\pi z}{t_2}-\frac{1}{n\pi}\left(E_0+\frac{E_1t_1}{2}\right)\times\left(1+\cos\frac{n\pi z}{t_2}\right)\right\}$$

$$\sigma_z=\sum_1^\infty\left(\phi_2e^{-\phi_2x}-\phi_1e^{-\phi_1x}\right)\frac{\phi_1\phi_2}{\phi_1-\phi_2}\left\{-\frac{4t_1t_2}{\pi^3n^3}\times\left(E_0+\frac{E_1t_1}{2}\right)\sin\frac{n\pi z}{t_2}+\frac{4t_1}{n^2\pi^2}\left(E_0+\frac{E_1t_1}{2}\right)\right.$$

$$\times \left[ \frac{t_1 t_2}{2} \left( 1 - \cos \frac{n\pi z}{t_2} \right) - t_1 + z \cos n\pi \right]$$

$$+ E_1 t_1 \frac{4 t_1^2}{\pi^4 n^4} \left( \cos \frac{n\pi z}{t_2} - \cos n\pi \right) \bigg) \bigg\} \tag{27b}$$

These expressions satisfy the governing equation (9) in region 1, all of the boundary conditions, and stress continuity at the interface between regions 1 and 2. However, because of the form of Eqs. (16), $\phi_1$ and $\phi_2$ are dependent on $t_1$ since $\lambda$ and $\mu$ were chosen to equal $n\pi z/t_1$. Thus, the solution in Eqs. (27) will be valid in region 2 only if $t_1 = t_2$ and regions 1 and 2 have the same compliances $S_{ij}$. If this restriction is not satisfied, the solution with $\phi_1$ and $\phi_2$ given by Eqs. (16) will not satisfy Eq. (9) in region 2. Alternatively, if $\phi_1$ and $\phi_2$ are chosen in region 2 so as to satisfy Eqs. (16), they will be different from $\phi_1$ and $\phi_2$ in region 1, and even though Eq. (9) will be satisfied in both regions, stress continuity at the interface between the two regions will not be satisfied.

This can be very tedious. As an approximation, only the first term in the series in Eqs. (27) is used. This will be quite accurate since examination of Eqs. (27) shows that, for $\tau_{xz}$ and $\sigma_z$, the terms in the series are proportional to $1/n^2$ and for odd $n$ the contribution of higher terms in the series will be negligible. Differentiating $\Pi_1$ with respect to $\phi_1$ and $\phi_2$ and setting the resulting expression individually equal to zero the following equations are obtained for $\phi_1$ and $\phi_2$:

$$\begin{aligned} &-(\phi_1+\phi_2)^2(R_1+R_2) - \phi_1^2 R_1 + \phi_1^3 \phi_2^2(\phi_1 + 2\phi_2) R_3 \\ &\quad + \phi_1^2 \phi_2^2 R_4 = 0 \\ &-(\phi_1+\phi_2)^2(R_1+R_2) - \phi_2^2 R_1 + \phi_1^2 \phi_2^3(\phi_2 + 2\phi_1) R_3 \\ &\quad + \phi_1^2 \phi_2^2 R_4 = 0 \end{aligned} \tag{30}$$

where the coefficients $R_1$, $R_2$, $R_3$, and $R_4$ are given in the Appendix. Equations (30) possess cyclic symmetry with respect to $\phi_1$ and $\phi_2$ and are nonlinear. Still, with some manipulation, they can be solved in closed form. The result is

$$\phi_1 = \left(\frac{R_1}{R_3}\right)^{1/4} \left[ \frac{ \dfrac{R_4(\sqrt{R_1/R_3}) - 2R_2}{R_1+R_2} \pm \left\{ \dfrac{[R_4(\sqrt{R_1/R_3}) - 2R_2]^2}{(R_1+R_2)^2} - 4 \right\}^{1/2} }{2} \right]^{-1/2}$$

$$\phi_2 = \left(\frac{R_1}{R_3}\right)^{1/4} \left[ \frac{ \dfrac{R_4(\sqrt{R_1/R_3}) - 2R_2}{R_1+R_2} \pm \left\{ \dfrac{[R_4(\sqrt{R_1/R_3}) - 2R_2]^2}{(R_1+R_2)^2} - 4 \right\}^{1/2} }{2} \right]^{1/2} \tag{31}$$

Another drawback of Eqs. (27) is that displacement compatibility at the interface between the regions 1 and 2 is not satisfied. To rectify these problems, the stress expressions as given by Eqs. (27) are assumed valid but $\phi_1$ and $\phi_2$ are treated as unknowns. It is important to note that $\phi_1$ and $\phi_2$ are functions of $n$. To determine $\phi_1$ and $\phi_2$ the energy in regions 1 and 2 is minimized. This will result in expressions (27) satisfying exactly stress equilibrium, all stress boundary conditions, and stress continuity. Deflection (or strain) compatibility will be satisfied on an average (energy) sense.

## Energy Minimization

For the problem at hand, the complementary energy is minimized with respect to the unknowns $\phi_1$ and $\phi_2$. This is very similar to procedures suggested for free-edge problems by Kassapoglou and Lagace[9] and Kassapoglou.[10] In fact, the whole solution is similar to the approaches in these two references. The energy expression to be minimized is

$$\Pi_c = \sum_1^2 \left[ \tfrac{1}{2} \iiint \sigma^T S \sigma \, \mathrm{d}V - \iint_{A_\sigma} T^T \bar{u} \, \mathrm{d}A_\sigma \right] \tag{28}$$

where the summation is over regions 1 and 2 and $S$ is the compliance tensor for each region. The prescribed displacements $\bar{u}$ multiply the tractions at $x = 0$ in region 2. The contribution of this term is negligible. So the function to be minimized is

$$\Pi_1 = \sum_1^2 \left\{ \iiint \left[ \left( S_{11} - \frac{S_{12}^2}{S_{22}} \right) \sigma_x^2 + \left( S_{33} - \frac{S_{23}^2}{S_{22}} \right) \sigma_z^2 + S_{55} \tau_{xz}^2 \right. \right.$$

$$\left. \left. + 2 \left( S_{13} - \frac{S_{12} S_{23}}{S_{22}} \right) \sigma_x \sigma_z \right] \mathrm{d}x \, \mathrm{d}z \right\} \tag{29}$$

The procedure would amount to differentiating Eq. (29) with respect to $\phi_1$ and $\phi_2$ for all values of $n$ and solving the resulting system of $(n+1)/2$ algebraic nonlinear equations ($n$ odd).

The solutions can be complex. Only solutions with positive real parts are acceptable to avoid ever increasing interlaminar stresses for large $x$. The solution to the problem will be given as a first approximation by the first term in the series of Eqs. (27) with $\phi_1$ and $\phi_2$ given by Eqs. (31).

## Results

The solution described in the previous sections was compared to a boundary element solution for a simple problem. The case is shown in Fig. 3. Typical thermoset elastic constants were used:

$E_{11} = 18.9$ msi $= 130$ Gpa

$E_{22} = E_{33} = 1.7$ msi $= 11.7$ GPa

$G_{12} = G_{13} = 0.73$ msi $= 5.0$ GPa

$G_{23} = 0.56$ msi $= 3.9$ GPa

$\nu_{12} = \nu_{13} = 0.29$

$\nu_{23} = 0.5$

The boundary element mesh is shown in Fig. 4. The two-dimensional code BEST2D developed at Pratt & Whitney was

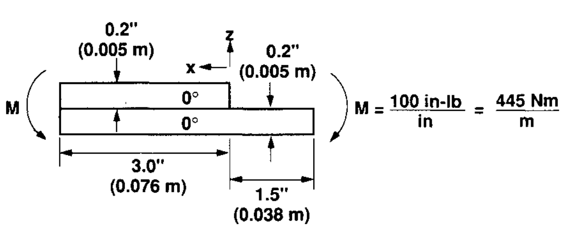


**Fig. 3 Sample problem.**

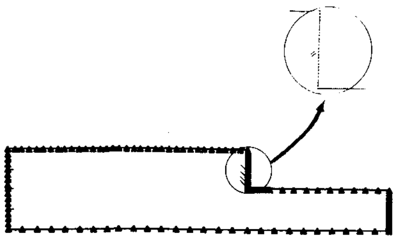

Fig. 4 Boundary element mesh.

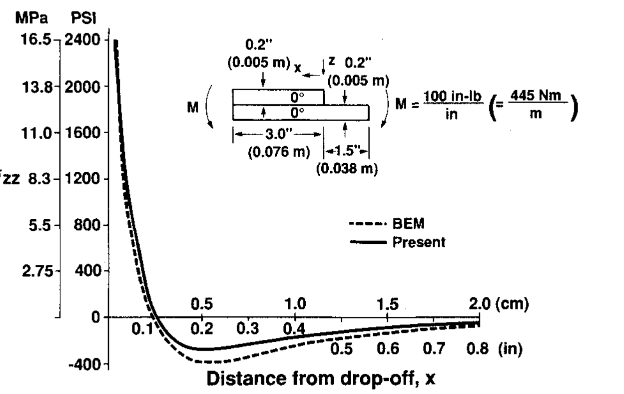


Fig. 5 Interlaminar normal stress at skin stiffener interface.

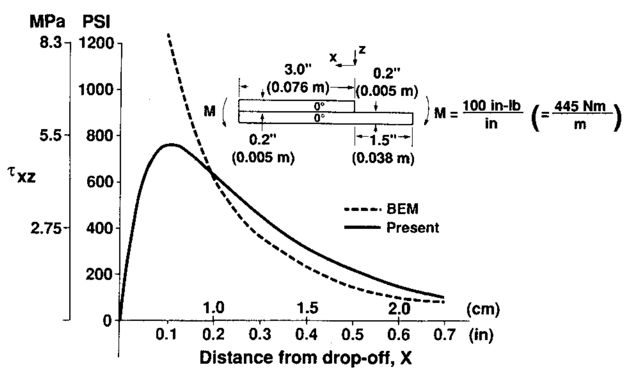


Fig. 6 Interlaminar shear stress at skin stiffener interface.

used. Boundary elements were used in this example because of the simplicity in generating a model and obtaining results over a finite element model. Previous comparisons for similar gradient problems had shown the boundary element results to be in excellent agreement with finite element solutions.

The results are shown in Figs. 5 and 6. Very good agreement between the present method and the boundary element solution is observed for the interlaminar normal stress $\sigma_z$ (Fig. 5). The small discrepancy for large $x$ values is attributed to the fact that only the first term in the series expansion is used. The agreement for the interlaminar shear stress $\tau_{xz}$ (Fig. 6) is reasonable except close to the edge where the imposed zero stress boundary condition in the present method results in differences from the boundary element solution that does not impose that condition.

The normal stress $\sigma_z$ at the edge itself appears to be singular. This is predicted by the present solution [see Eqs. (27)] where the series expression for $\sigma_z$ diverges when $x=0$. The boundary element predictions seem to verify this since the stress at the edge appeared to increase without bound as the mesh was made finer. The presence of a weak singularity in such edge problems has been documented by, among others, Wang and Choi.[4]

## Conclusions

The present solution can be improved significantly by including more terms in the series in Eqs. (27) and solving the enlarged system of Eqs. (30). Also, the far-field in-plane stress can be different from linear. If the far-field stress can be expanded in a Fourier series, the method can be applied with proper modification of the series coefficients. Because the method is so efficient and simple, the solution can be easily applied to various designs and lay ups (where the plies in the flange and skin are smeared much like a substructure) to generate stress plots at the skin/stiffener interface and help determine early in the design stage if interlaminar stresses will be a problem (this is in conjunction with a delamination onset criterion such as the one proposed by Brewer and Lagace[11]). It should be pointed out that the current method is to be used for preliminary design. For final analysis, more detailed methods may be necessary.

Finally, Eqs. (9) and (11) are quite general and apply to the whole family of problems shown in Fig. 1. It is expected that similar procedures can be developed for the remaining problems in Fig. 1 even though, for some of them, satisfying the boundary conditions may be quite involved.

In summary, a simple efficient method to determine interlaminar stresses at the skin/stiffener interface of composite stiffened panels has been presented. The solution is in good agreement with more detailed boundary element solutions and can be used to evaluate the candidate designs and point to potential delamination problems. The method is sufficiently general so that, with some modifications, it can be applied to other similar boundary-layer problems in composites.

## Appendix: Equation Coefficients

The coefficients in the two equations for $\phi_1$ and $\phi_2$ are given in the following.

$$R_1 = F_{11} + G_{11}$$

$$R_2 = F_{12} + G_{12}$$

$$R_3 = F_2 + G_2$$

$$R_4 = F_3 + F_4 + G_3 + G_4$$

$$F_{11} = \left(S_{11} - \frac{S_{12}^2}{S_{22}}\right)\frac{4t_1}{\pi^2}\left[\left(E_0 + \frac{E_1 t_1}{2}\right)^2 + \frac{(E_1 t_1)^2}{\pi^2}\right]$$

$$F_{12} = -\left(S_{11} - \frac{S_{12}^2}{S_{22}}\right)\left\{\frac{16t_1}{\pi^2} E_0\left(E_0 + \frac{E_1 t_1}{2}\right) + \frac{8t_1^2 E_1}{\pi^2}\right.$$

$$\left. \times \left[\frac{2E_1 t_1}{\pi^2} + \left(E_0 + \frac{E_1 t_1}{2}\right)\right]\right\}$$

$$F_2 = -\left(S_{33} - \frac{S_{23}^2}{S_{22}}\right)\frac{8t_1^5}{\pi^6}\left[\left(E_0 + \frac{E_1 t_1}{2}\right)^2 + \left(\frac{\pi^2}{3} - \frac{3}{2}\right) + \frac{3}{2\pi^2}\right.$$

$$\left. \times (E_1 t_1)^2 - E_1 t_1\left(E_0 + \frac{E_1 t_1}{2}\right)\right]$$

$$F_3 = S_{55}\frac{8t_1^3}{\pi^4}\left[\frac{3}{2}\left(E_0 + \frac{E_1 t_1}{2}\right)^2 \frac{(E_1 t_1)^2}{2\pi^2} - \frac{4E_1 t_1}{\pi^2}\left(E_0 + \frac{E_1 t_1}{2}\right)\right]$$

$$F_4 = -2\left(S_{13} - \frac{S_{12}S_{23}}{S_{22}}\frac{8t_1^3}{\pi^4}\left[\frac{1}{2}\left(E_0 + \frac{E_1 t_1}{2}\right)^2 - \frac{(E_1 t_1)^2}{2\pi^2}\right]\right.$$

$$G_{11} = \left(S_{11} - \frac{S_{12}^2}{S_{22}}\right)\frac{4t_1^2}{t_2^2}\left\{\frac{t_2}{\pi^2}\left(E_0 + \frac{E_1 t_1}{2}\right)^2 + \frac{1}{t_2}\left[\frac{(t_1 + t_2)^2}{4}\right.\right.$$

$$\times\left(E_0+\frac{E_1t_1}{2}\right)^2+\frac{t_1^2(E_1t_1)^2}{\pi^4}-(t_1+t_2)$$

$$\times\left(E_0+\frac{E_1t_1}{2}\right)(E_1t_1)\,\frac{t_1}{\pi^2}\bigg]\bigg\}$$

$$G_{12}=\left(S_{11}-\frac{S_{12}^2}{S_{22}}\right)\frac{8t_1}{\pi^2}\left\{2\bar{E}_0\left(E_0+\frac{E_1t_1}{2}\right)+\bar{E}_1\right.$$

$$\left.\times\left[-t_1\left(E_0+\frac{E_1t_1}{2}\right)\frac{2t_1}{\pi^2}E_1t_1\right]\right\}$$

$$G_2=\left(S_{33}-\frac{S_{23}^2}{S_{22}}\right)\left[-12\,\frac{t_1^2t_2^3}{\pi^6}\left(E_0+\frac{E_1t_1}{2}\right)^2\right.$$

$$+\frac{8t_1^2}{\pi^4}\left(E_0+\frac{E_1t_1}{2}\right)^2\left(\frac{3}{8}\,t_1^2t_2+\frac{1}{4}\,t_1t_2^2-\frac{5}{24}\,t_2^3\right)$$

$$\left.+12\,\frac{t_1^4t_2}{\pi^8}(E_1t_1)^2-\frac{4}{\pi^6}\,t_1^3t_2\,(E_1t_1)\left(E_0+\frac{E_1t_1}{2}\right)(3t_1+t_2)\right]$$

$$G_3=S_{55}\,\frac{8t_1^2}{\pi^2}\left\{\frac{1}{2t_2}\left[\frac{t_1t_2}{2}\left(E_0+\frac{E_1t_1}{2}\right)-\frac{t_1}{\pi^2}E_1t_1\right]^2+\frac{3}{2}\,\frac{t_2}{\pi^2}\right.$$

$$\times\left(E_0+\frac{E_1t_1}{2}\right)^2-\frac{4}{\pi^2}\left[\frac{t_1t_2}{2}\left(E_0+\frac{E_1t_1}{2}\right)\right.$$

$$\left.\left.-\frac{t_1}{\pi^2}E_1t_1\right]\left(E_0+\frac{E_1t_1}{2}\right)\right\}$$

$$G_4=-2\left(S_{13}-\frac{S_{12}S_{23}}{S_{22}}\right)\frac{2t_1}{t_2}\left[-\left(E_0+\frac{E_1t_1}{2}\right)^2\frac{t_1}{2\pi^2}(t_1+t_2)^2\right.$$

$$+2\left(E_0+\frac{E_1t_1}{2}\right)E_1t_1\frac{t_1^2(t_1+t_2)}{\pi^4}-(E_1t_1)^2\,\frac{2t_1^3}{\pi^6}$$

$$\left.+\left(E_0+\frac{E_1t_1}{2}\right)^2\frac{2t_1t_2^2}{\pi^4}\right]$$

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